ADVANCED MATHEMATICS

Final Exam - June 2014

SOLUTIONS

Instructions: This exam consists of six questions. You have two hours to give a reasoned answer to all the exercises. Write the quiz entirely in ink. Calculators are not permitted.

1 Determine the values of the parameter $a \in \mathbb{R}$ for which the matrix A is diagonalizable.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 1 \\ -3 & 0 & 3 \end{pmatrix}$$

Solution. The characteristic polynomial is $p(\lambda) = |A - \lambda I| = (1-\lambda)(a-\lambda)(3-\lambda)$. Thus, the eigenvalues of A are $\sigma(A) = \{1, a, 3\}$. If $a \notin \{1, 3\}$ then the roots of the characteristic polynomial are different each other, and hence, A can be diagonalized.

(a) If $a = 1$. Then $\sigma(A) = \{1, 3\}$ with $m(1) = 2$ and $m(3) = 1$. On the other hand, $\dim S(1) = 3 - (A - 1 \cdot I_3) = 3 - 2 = 1$. Since the dimension of the eigenspace and the multiplicity do not coincide, the matrix A is not diagonalizable when $a = 1$.

(b) If $a = 3$. Then $\sigma(A) = \{1, 3\}$ with $m(1) = 1$ and $m(3) = 2$. On the other hand, $\dim S(3) = 3 - (A - 3 \cdot I_3) = 3 - 2 = 1$. Since the dimension of the eigenspace and the multiplicity do not coincide, the matrix A is not diagonalizable when $a = 3$.

2 Suppose that there is market with a single commodity whose demand function is $D(P) = 4 - 2P$ and whose supply function is $S(P) = -2 + P$, where $P > 0$ denotes the unit price of the commodity. Assume that time is discrete and the market behaves according to the cobweb model, that is, $S(P_t) = D(P_{t+1})$ for each t.

(a) Obtain the expression of P_t when $P_0 = 4$.

(b) Obtain the equilibrium $\overline{P}$.

(c) Analyze the behavior of the price in the long run.

Solution.

(a) The difference equation we need to solve is

$$-2 + P_t = 4 - 2P_{t+1} \quad \equiv \quad 2P_{t+1} + P_t = 6$$

The characteristic polynomial of the associated homogeneous equation is $2r + 1 = 0$; hence $P_t^h = A_0\left(-\frac{1}{2}\right)^t$. Since the independent term $b_t = 6$ is a polynomial of degree zero, we try as particular solution $P_t = C$. Once we substitute into the equation we obtain that $P_t^p = C = 2$. Finally $P_t = A_0\left(-\frac{1}{2}\right)^t + 2$. Since $P_0 = 4$ (and then $A_0 = 2$), we conclude that

$$P_t = 2\left(-\frac{1}{2}\right)^t + 2.$$

(b) The equilibrium is a value $\overline{P}$ such that $\overline{P} + 2\overline{P} = 6$, this is, $\overline{P} = 2$.

(c) In the long run, $\lim_{t\to\infty} P_t = 2$. So, it converges to the equilibrium.

3 Consider the following system of equations

$$X_{t+1} = \begin{pmatrix} 2 & 0 \\ 3 & -2 \end{pmatrix} X_t.$$

(a) Obtain the solutions of the previous system.

(b) Compute the equilibrium $\overline{X}$.

(c) Is the equilibrium $\overline{X}$ globally asymptotically stable?

Solution.

(a) The eigenvalues are $_1 = 2$ and $\lambda_2 = -2$, with corresponding eigenvectors $u_1 = (4,3)$ and $u_2 = (0,1)$. Therefore, the solution is:

$$X_t = A_1 2^t \begin{pmatrix} 4 \\ 3 \end{pmatrix} + A_2(-2)^t \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(b) The equilibrium is the point $\overline{X}$ such that $\overline{X} = A\overline{X}$. Then, $\overline{X}^T = (0,0)$.

(c) Since $|2| > 1$, the equilibrium are not globally asymptotically stable.

4 Solve the following differential equation: $t^2x' = 1 - tx$.

Solution. This is linear equation whose canonical form is $x' + \frac{1}{t}x = \frac{1}{t^2}$. We compute $\mu(t) = e^{\int \frac{1}{t}dt} = t$. Mutiplying both sides by $\mu(t)$ and making some transformation we get that $(x \cdot t)' = \frac{1}{t}$. Therefore $x(t) = \frac{\ln t}{t} + \frac{C}{t}$.

5 Solve the following equation: $x'' - 4x' + 4x = \sin t$.

Solution. The root of the characteristic polynomial is $r = 2$ with multiplicity 2. Then,

$$x^h(t) = A_0 e^{2t} + A_1 t e^{2t}.$$

Since $b(t) = \sin t$, we propose $x^p(t) = C_0 \sin t + D_0 \cos t$. Taking the need derivatives and substituting we get that $C_0 = \frac{3}{7}$ and $D_0 = \frac{4}{7}$. And then, $x^p(t) = \frac{3}{7}\sin t + \frac{4}{7}\cos t$.Finally

$$x(t) = A_0 e^{2t} + A_1 t e^{2t} + \frac{3}{7}\sin t + \frac{4}{7}\cos t$$

6 Consider the differential equation $x' = f(x)$. The following picture shows the trajectories of the solution of this equation.

(a) Identify the equilibria.

(b) Study the stability.

(c) Make a sketch of the phase diagram corresponding to this situation.

Solution.

(a) The equilibria are $\overline{x} = -2$, $\overline{x} = 2$, and $\overline{x} = 6$.

(b) $\overline{x} = -2$ and $\overline{x} = 6$ are unstable, while $\overline{x} = 2$ is stable.

(c) The sketch is